

# Quantum signal+noise models: beyond i.i.d.\*

A. S. Holevo

Steklov Mathematical Institute  
of Russian Academy of Sciences, Moscow

## Abstract

Recently, the Gaussian optimizer conjecture in quantum information theory was confirmed for bosonic Gaussian gauge-covariant or contravariant channels. These results use the *i.i.d.* model of the quantum noise.

In this paper we consider quantum Gaussian signal+noise model with *time-continuous stationary coloured noise*. A proof of the coding theorem for the classical capacity of quantum broadband gauge-covariant Gaussian channels is proposed. We also discuss and compare the “broadband” and the “bandpass” models of time-continuous time-continuous stationary coloured noise.

## 1 Introduction

Recently, the Gaussian optimizer conjecture in quantum information theory was confirmed for bosonic Gaussian gauge-covariant or contravariant channels including phase-insensitive channels such as attenuators, amplifiers and additive classical noise channels [5]. It is shown that the classical capacity of these channels under the input energy constraint is additive and achieved by Gaussian encodings. These results use the *i.i.d.* model of quantum noise.

In this paper we consider a quantum Gaussian signal+noise model with *time-continuous stationary coloured noise*. In this context we propose a proof of coding theorem for the classical capacity of quantum broadband gauge-covariant Gaussian channels. We also discuss and compare the “broadband”

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and the “bandpass” models of time-continuous stationary noise. It is well-known that in the classical case a rigorous treatment of Gaussian channels with time-continuous stationary coloured noise requires some advanced mathematical tools such as the spectral theory for integral operators with continuous symmetric kernels and Kac-Merddock-Szöge theorem (see ch. 8 of [4]). The present paper makes a step towards achieving similar goal in the quantum case, where additional difficulties due to the symplectic structure related to commutation properties of observed processes arise.

There were several previous works where such problems were considered for different special cases, with different degree of justification. In this paper we rely upon the proof of coding theorem for a model “classical signal+quantum Gaussian noise” involving the Planck spectrum, given in [10]. In the paper of V. Giovannetti, S. Lloyd, L. Maccone, P.W. Shor [6] the authors considered a broadband pure-loss channel by formal passage from discrete to continuous spectrum and demonstrated numerical solutions for the capacities  $C$ ,  $C_{ea}$ ,  $Q$ . The paper of G. De Palma, A. Mari, V. Giovannetti [3] was devoted to a rigorous treatment of discrete time, Markov memory model, with flat noise spectrum. Recently B. R. Bardhan, J. H. Shapiro [1] studied a narrowband approximation for phase-insensitive time-invariant channels using the result of [5].

The classical AGWN model is given by the equation

$$Y_k = X_k + Z_k; \quad k = 1, \dots, n \quad (1)$$

where  $Z_k \sim \mathcal{N}(0, N)$  are real Gaussian i.i.d. random variables representing the noise and the signal sequence  $X_k$  is subject to the energy constraint

$$n^{-1}(X_1^2 + \dots + X_n^2) \leq E.$$

The asymptotic ( $n \rightarrow \infty$ ) capacity of this model is given by the famous Shannon formula<sup>1</sup>

$$C = \frac{1}{2} \log(E + N) - \frac{1}{2} \log N = \frac{1}{2} \log \left( 1 + \frac{E}{N} \right). \quad (2)$$

In the quantum analog of the signal+noise equation

$$Y = X + Z$$

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<sup>1</sup>Throughout this paper we use natural logarithms. In the context of quantum channels “capacity” will always mean the *classical capacity*.

one replaces the classical variables  $X, Y, Z$  by a (multiple of) pair of selfadjoint operators  $q, p$ , satisfying the Heisenberg canonical commutation relation (CCR)  $[q, p] = i\hbar I$ , or, equivalently, by a single operator  $a = \frac{1}{\sqrt{2\hbar\omega}} (\omega q + ip)$  (with Hermitean conjugate  $a^\dagger = \frac{1}{\sqrt{2\hbar\omega}} (\omega q - ip)$ ), satisfying the canonical commutation relation (CCR)

$$[a, a^\dagger] = I. \quad (3)$$

In applications  $p$  and  $q$  describe quantized quadratures of the harmonic mode of frequency  $\omega$ ,

$$q\omega \cos \omega t + p \sin \omega t = \sqrt{\frac{\hbar\omega}{2}} (ae^{-i\omega t} + a^\dagger e^{i\omega t}) \quad (4)$$

while  $a, a^\dagger$  are quantizations of the complex amplitude and its adjoint.

As distinct from the classical case, the quantum models should respect the CCR i.e. arise as a part of a linear canonical transformation. Below we give a list of such models which have additional property of gauge symmetry to be explained later. In these models  $a$  presents the quantum input signal,  $b$  – the quantum Gaussian noise variable and  $a'$  – the quantum output signal, all of them satisfying the CCR (3).

1. Attenuator

$$a' = ka + \sqrt{1 - k^2}b, \quad 0 \leq k \leq 1.$$

2. Amplifier

$$a' = ka + \sqrt{k^2 - 1}b^\dagger, \quad k \geq 1.$$

3. Additive classical Gaussian noise

$$a' = a + \eta,$$

where  $\eta$  is the classical complex random variable having circular Gaussian distribution.

4. Phase-invertive amplifier

$$a' = ka^\dagger + \sqrt{k^2 + 1}b, \quad k \geq 0.$$

5. Classical-quantum channel (state preparation)

$$a' = x + b,$$

where  $x$  is the complex random variable representing classical signal at the background quantum Gaussian noise  $b$ .

6. Quantum-classical channel (heterodyne measurement)

$$y = a + b^\dagger,$$

where  $y$  is the complex random variable representing the classical output of the measurement,  $a$  is quantum signal,  $b$  – quantum Gaussian noise.

All these equations have the form  $Y = X + Z$ , where the noise  $Z$  is described by quantum or classical variable in Gaussian state<sup>2</sup> with the first two moments determined by

$$\langle Z \rangle = 0, \quad \langle Z^\dagger Z \rangle = N, \quad \langle Z Z \rangle = 0. \quad (5)$$

Thus in all the cases 1-6 the quantum AGWN model has the form (1) where  $Z_k$  are quantum or classical Gaussian i.i.d. noise variables obeying (5) and the signal sequence  $X_k$  is subject to the energy constraint

$$n^{-1} \langle X_1^\dagger X_1 + \dots + X_n^\dagger X_n \rangle \leq E.$$

A basic difference of the quantum signal variables is that one cannot simply impose on them zero or other deterministic values; one should instead define the *state* describing these variables (in the classical case the deterministic values are obtained from degenerate probability distributions).

This circumstance underlies a basic difficulty in finding the quantum analog of the Shannon formula (2): finding the minimum of the output entropy in the formula

$$C_1 = \max_{\langle X^\dagger X \rangle \leq E} H(Y) - \min_X H(Y).$$

Another problem is the proof of additivity of “ $n$ -shot capacity”,  $C_n = nC_1$ . When the signal  $X$  is classical (case 5), the minimum  $\min_X H(Y) = H(Z)$

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<sup>2</sup>For detailed account of quantum Gaussian states see [12], [14].

is attained for  $X \equiv 0$ . The resulting solution for the asymptotic capacity obtained in [10] is

$$C = g(E + N) - g(N), \quad (6)$$

where

$$g(N) = (N + 1) \log(N + 1) - N \log N \quad (7)$$

is the function representing the entropy of quantum Gaussian state with the moments (5).

The capacity of heterodyne measurement (case 6) was obtained in [7] by using a special “information-exclusion” method and is equal to

$$C = \log(E + N) - \log N. \quad (8)$$

Alternatively, the minimal output entropy can be found using Lieb’s solution of Wehrl’s conjecture [13] saying that the minimum is attained on the coherent states.

In the cases 1-4 a similar “Gaussian optimizers conjecture” [16] was open for a dozen of years and finally solved in [5] (see Appendix 1). The resulting capacity formula in the cases 1-3 has the same form as (6), i.e.

$$C = g(E + N) - g(N), \quad E = \langle X^\dagger X \rangle, \quad N = \langle Z^\dagger Z \rangle, \quad (9)$$

while in the case 4 it is

$$C = g(E + N) - g(N + k^2). \quad (10)$$

All these solvable models possess symmetry under the gauge transformation  $a \rightarrow a e^{i\varphi}$ ,  $\varphi \in \mathbb{R}$ . The quantum channels 1-3, as well as “hybrid” channels 5 (classical-quantum) and 6 (quantum-classical) are gauge-covariant i.e. their output changes similarly to the input:  $a' \rightarrow a' e^{i\varphi}$ , while the channel 4 is gauge-contravariant:  $a' \rightarrow a' e^{-i\varphi}$ . A complete classification of normal forms of single-mode quantum Gaussian channels was given in [11]. In this classification the cases 1-4 represent those normal forms which possess the gauge symmetry, while 5,6 are the hybrid cases with this symmetry.

In the classical prototype of the gauge-covariant models  $X, Y, Z$  are complex Gaussian random variables having circular distribution and the capacity is twice the Shannon expression (2) i.e.  $C = \log(E + N) - \log N$ .

## 2 The coding theorem for a broadband quantum channel

In classical information theory the broadband channel can be treated by reduction to parallel channels, i.e. by decomposing the Gaussian stochastic process into independent one-dimensional harmonic modes (4). In quantum theory such a decomposition plays an important additional role as a tool for *quantization* of the classical process. As a starting point for the time-domain model of quantum noise we take the expression for quantized electric field in a square box of size  $L$  (see, e.g. [8])

$$E(\mathbf{x}, t) = \frac{i}{L^{3/2}} \sum_k \sqrt{\frac{\hbar \omega_k}{2}} a_k e^{i\mathbf{k}\mathbf{x}} e^{-i\omega_k t} + \text{h.c.}$$

where  $a_k^\dagger, a_k$  are the creation-annihilation operators of independent bosonic modes satisfying the standard canonical commutation relations<sup>3</sup>

$$[a_j, a_k^\dagger] = \delta_{jk} I, \quad [a_j, a_k] = 0. \quad (11)$$

Basing on this expression and redefining  $a_k$ , we consider the following periodic operator-valued function as a model for observations on the time interval  $[0, T]$  at the spatial point  $\mathbf{x} = \mathbf{0}$ :

$$\hat{Z}(t) = \sum_k \sqrt{\frac{\hbar \omega_k}{2T}} \left( a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t} \right), \quad t \in [0, T], \quad (12)$$

$$\omega_k = \frac{2\pi k}{T}, \quad k = 1, 2, \dots; \quad \Delta\omega = \frac{2\pi}{T},$$

see [10]. To avoid ultraviolet divergence, we introduce the cutoff function  $\bar{\omega}(T), T > 0$ , with the properties:  $\bar{\omega}(T)$  is positive and monotonously increasing with  $\lim_{T \rightarrow \infty} \bar{\omega}(T) = \infty$ , and for each  $T$  include in all summations over  $k$  only the frequencies  $\omega_k \in [0, \bar{\omega}(T)]$ . Then the energy operator has the expression (as distinct from the narrowband approximation):

$$\int_0^T \hat{Z}(t)^2 dt = \sum_k \hbar \omega_k \left( a_k^\dagger a_k + \frac{1}{2} \right)$$

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<sup>3</sup>For simplicity we do not consider the polarization degree of freedom.

We modify the argument of [10] related to classical-quantum channel and generalize it to include the Gaussian gauge-covariant channels (cases 1-3). For a fixed  $T$  the “in-out” equations of the channel  $\Phi_T$  for the collection of frequency modes are

$$a_{k,Y} = K(\omega_k) a_{k,X} + \hat{n}_{k,Z}, \quad 0 \leq \omega_k \leq \bar{\omega}(T), \quad (13)$$

with the noise operators

$$\hat{n}_{k,Z} = \begin{cases} \sqrt{1 - |K(\omega_k)|^2} a_{k,Z}, & |K(\omega_k)| < 1 \quad \text{attenuator} \\ \eta_{k,Z}, & |K(\omega_k)| = 1 \quad \text{class.noise} \\ \sqrt{|K(\omega_k)|^2 - 1} a_{k,Z}^\dagger(\omega) & |K(\omega_k)| > 1 \quad \text{amplifier} \end{cases}$$

satisfying the commutation relations

$$[\hat{n}_{k,Z}, \hat{n}_{l,Z}^\dagger] = \delta_{kl} (1 - |K(\omega_k)|^2),$$

and described by a centered Gaussian state with the second moments

$$\langle \hat{n}_{l,Z}^\dagger \hat{n}_{k,Z} \rangle = \delta_{kl} N(\omega_k), \quad \langle \hat{n}_{l,Z} \hat{n}_{k,Z} \rangle = 0.$$

Here  $K(\omega)$ ,  $N(\omega)$  are continuous functions,  $N(\omega) \geq 0$  in the domain  $\omega \geq 0$ .

Then  $\Phi_T$  is Gaussian gauge-covariant channel in the Hilbert space  $\mathcal{H}_T$  of the modes with frequencies  $0 \leq \omega_k \leq \bar{\omega}(T)$ , whose action on the quantum states (density operators in  $\mathcal{H}_T$ ) is described in [11], see also Ch. 12 of [14]. We consider the family  $\{\Phi_T; T \rightarrow \infty\}$  as our model for the broadband channel.

**Definition.** For each  $T > 0$  a *code*  $(\Sigma, M)$  is a collection  $\{\rho^j, M_j; j = 1 \dots N\}$  where  $\rho^j$  are quantum states in  $\mathcal{H}_T$  satisfying the energy constraint<sup>4</sup>

$$\text{Tr } \rho^j \left( \sum_k \hbar \omega_k a_{k,X}^\dagger a_{k,X} \right) \leq ET, \quad (14)$$

and  $M$  is a POVM in  $\mathcal{H}_T$ .

We define the *capacity* of the family  $\{\Phi_T; T \rightarrow \infty\}$  as the supremum of rates  $R$  for which the infimum of the average error probability

$$\bar{\lambda}_T(\Sigma, M) = \frac{1}{N} \sum_{j=1}^N (1 - \text{Tr } \Phi_T[\rho^j] M_j).$$

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<sup>4</sup>Notice that the vacuum energy  $\frac{1}{2} \sum_k \hbar \omega_k$  is explicitly excluded from the constraint to avoid the divergence when  $T \rightarrow \infty$ .

with respect to all codes of the size  $N = e^{TR}$  tends to zero as  $T \rightarrow \infty$ .

**Theorem.** *Let  $N(\omega), K(\omega)$  be continuous functions,  $0 < |K(\omega)| \leq \kappa$ , and  $\bar{\omega}(T)/T \rightarrow 0$  as  $T \rightarrow \infty$ . The capacity of the family of channels  $\{\Phi_T; T \rightarrow \infty\}$  is equal to*

$$C = \int_0^\infty (g(\tilde{N}_\theta(\omega)) - g(N(\omega)))_+ \frac{d\omega}{2\pi}, \quad (15)$$

where

$$\tilde{N}_\theta(\omega) = \frac{1}{e^{\theta \hbar \omega / |K(\omega)|^2} - 1},$$

and  $\theta$  is chosen such that

$$\int_0^\infty (\hbar \omega / |K(\omega)|^2) (\tilde{N}_\theta(\omega) - N(\omega))_+ \frac{d\omega}{2\pi} = E.$$

The capacity is upperbounded as

$$C \leq \frac{\pi \kappa^2}{6 \hbar \theta}.$$

The proof given in the Appendix 2 combines the solution of the quantum Gaussian optimizer conjecture [5] with the estimates from the proof of the coding theorem for constrained infinite dimensional channel [10]. The underlying mechanism is emergence of increasing number of parallel independent channels in arbitrarily small neighbourhood of each frequency. Similar proof applies to the classical capacities of time-domain versions of gauge-contravariant channel (10) resulting in:

$$C = \int_0^\infty (g(\tilde{N}_\theta(\omega)) - g(N(\omega) + |K(\omega)|^2))_+ \frac{d\omega}{2\pi}.$$

For the case of quantum-classical channel (8) one has

$$C = \int_0^\infty (\log N_\theta(\omega) - \log N(\omega))_+ \frac{d\omega}{2\pi},$$

where  $N_\theta(\omega)$  is given by (16) below.

For completeness we briefly recall here the case of classical-quantum channel which was considered in [10]. The channel equation in the frequency domain is:

$$\Phi_T : \quad a_{k,Y} = x_k + a_{k,Z}.$$



In this case it can be rewritten in the time domain as “classical signal + quantum noise” equation

$$\hat{Y}(t) = X(t) + \hat{Z}(t), \quad t \in [-T/2, T/2],$$

where the classical signal

$$X(t) = \sum_k \sqrt{\frac{\hbar\omega_k}{2T}} (x_k e^{-i\omega_k t} + \bar{x}_k e^{i\omega_k t}), \quad x_k \in \mathbb{C}.$$

The mean power constraint on the signal

$$\sum_k \hbar\omega_k |x_k|^2 = \int_0^T X(t)^2 dt \leq ET.$$

Then with appropriate modification of Definition of the code, one obtains the expression for the classical capacity

$$C = \int_0^\infty (g(N_\theta(\omega)) - g(N(\omega)))_+ \frac{d\omega}{2\pi},$$

$$N_\theta(\omega) = \frac{1}{e^{\theta\hbar\omega} - 1}, \tag{16}$$

and  $\theta$  is chosen such that

$$\int_0^\infty \hbar\omega (N_\theta(\omega) - N(\omega))_+ \frac{d\omega}{2\pi} = E.$$

which coincides with the expression (15) for  $K(\omega) \equiv 1$ .

An example is the case of equilibrium quantum noise  $N(\omega) = N_{\theta_P}(\omega) \equiv (e^{\theta_P\hbar\omega} - 1)^{-1}$  with  $\theta_P = \sqrt{\pi/12\hbar P}$  determined from

$$\int_0^\infty \frac{\hbar\omega}{e^{\theta_P\hbar\omega} - 1} \frac{d\omega}{2\pi} = P.$$

Then

$$\int_0^\infty g((e^{\theta_P\hbar\omega} - 1)^{-1}) = \frac{\pi}{6\hbar\theta_P} = \sqrt{\frac{\pi P}{3\hbar}}$$

(see e.g. [10] for detail of computation) and

$$C = \sqrt{\frac{\pi(P+E)}{3\hbar}} - \sqrt{\frac{\pi P}{3\hbar}},$$

which is similar to the capacity of the semiclassical broadband *photonic channel* [17], [2].

## 3 Discussion

### 3.1 The limiting broadband noise model

In the limit  $T \rightarrow \infty$  of the periodic process (12) converges in distribution to the quantum stationary Gaussian noise [10], [9]

$$\hat{Z}(t) = \int_0^\infty \sqrt{\frac{\hbar\omega}{2}} \left( d\hat{A}(\omega)e^{-i\omega t} + d\hat{A}(\omega)^\dagger e^{i\omega t} \right).$$

Here  $\hat{A}(\omega)$ ,  $\omega \geq 0$ , is the quantum Gaussian independent increment process with the commutators

$$[d\hat{A}(\omega), d\hat{A}(\omega')^\dagger] = \frac{1}{2\pi} \delta(\omega - \omega') d\omega d\omega', \quad [d\hat{A}(\omega), d\hat{A}(\omega')] = 0, \quad (17)$$

zero mean, and the normally-ordered correlation

$$\langle d\hat{A}(\omega)^\dagger d\hat{A}(\omega') \rangle = \frac{1}{2\pi} \delta(\omega - \omega') N(\omega) d\omega d\omega'. \quad (18)$$

This can be considered as an inhomogeneous generalization of the quantum Brownian motion of Hudson-Parthasarathy [19], albeit in the frequency domain.

The noise commutator is causal

$$[\hat{Z}(t), \hat{Z}(s)] = i\hbar/2 \int_0^\infty \omega \sin \omega(s-t) d\omega = i\hbar/2 \delta'(t-s),$$

and the noise symmetrized correlation function is

$$\alpha(t-s) \equiv \langle \hat{Z}(t) \circ \hat{Z}(s) \rangle = \beta(t-s) + \frac{1}{2}j(t-s),$$

where

$$\begin{aligned} \beta(t) &= \hbar \int_0^\infty \omega N(\omega) \cos \omega t \frac{d\omega}{2\pi}, \\ j(t) &= \hbar \int_0^\infty \omega \cos \omega t \frac{d\omega}{2\pi} = -\frac{\hbar}{2\pi} t^{-2}. \end{aligned}$$

so that the vacuum symmetrized correlation function  $\frac{1}{2}j(t-s)$ .

One can then introduce the gauge-covariant channels in the frequency domain by the equation

$$d\hat{A}_Y(\omega) = K(\omega)d\hat{A}_X(\omega) + d\hat{A}_Z(\omega), \quad (19)$$

where the appropriately modified Gaussian noise  $\hat{Z}(t)$  satisfies (cf. [1])

$$\begin{aligned} [d\hat{A}_Z(\omega), d\hat{A}_Z^\dagger(\omega')] &= \frac{1}{2\pi}\delta(\omega - \omega')(1 - |K(\omega)|^2)d\omega d\omega', \\ \langle d\hat{A}_Z^\dagger(\omega) d\hat{A}_Z(\omega') \rangle &= \frac{1}{2\pi}\delta(\omega - \omega')N(\omega)d\omega d\omega'. \end{aligned}$$

In the time domain, asymptotically (as  $T \rightarrow \infty$ )

$$\hat{Y}(t) \approx (KX)(t) + \hat{Z}(t), \quad (20)$$

with nonanticipating real-valued filter

$$(K\hat{X})(t) = \int_0^t \hat{X}(s)k(t-s)ds, \quad K(\omega) = \int_0^\infty k(t)e^{i\omega t}dt = \overline{K(-\omega)}.$$

If  $K$  is instantaneous or has finite memory, then (20) becomes equality.

The noise is *generalized* quantum (operator-valued) Gaussian process,  $R(f) = \int_{-\infty}^\infty \hat{Z}(t)f(t)dt$ , where  $f$  runs over an appropriate space of test functions. The mathematical construction which gives to it a rigorous meaning is based on quasi-free representations of the C\*-algebra  $\mathfrak{A}(\mathcal{H}, \Delta)$  of CCR [18] over the symplectic space  $\mathcal{H} = \mathcal{K}(\mathbb{R})$  of real-valued infinite differentiable functions with compact support, with the skew-symmetric form  $\Delta$ , and the vacuum inner product  $j$ , given by ( $\hbar = 2$ )

$$\begin{aligned} \Delta(f, g) &= \int_{-\infty}^\infty f(t)\frac{d}{dt}g(t)dt = \pi^{-1}\text{Im} \int_0^\infty \omega \overline{\tilde{f}(\omega)}\tilde{g}(\omega) d\omega, \\ j(f, g) &= \pi^{-1}\text{Re} \int_0^\infty \omega \overline{\tilde{f}(\omega)}\tilde{g}(\omega) d\omega \\ &= \pi^{-1} \int_{-\infty}^\infty g(t) \int_{-\infty}^\infty \frac{2f(t) - f(t-s) - f(t+s)}{s^2} ds dt \\ &= (2\pi)^{-1} \int_{-\infty}^\infty \int_{-\infty}^\infty (g(t) - g(t-s))(f(t) - f(t-s))s^{-2} ds dt, \end{aligned} \quad (21)$$

see n.7.1 of [9]. The operator of complex structure  $J$  is multiplication by  $i \operatorname{sgn}(\omega)$  in the frequency domain or the Hilbert transform in the time domain

$$(Jz)(t) = \frac{1}{\pi} \text{V.P.} \int_{-\infty}^{\infty} \frac{z(s)}{t-s} ds$$

Here  $H = \overline{K_{\infty}(\mathbb{R})}$ , the completion is with respect to Slobodeckij (semi)norm, corresponding to the inner product (21) [20]. Thus the relevant space is Sobolev-Slobodeckij space  $H^{1/2}(\mathbb{R})$  of half-differentiable functions.

A natural conjecture would be that the asymptotic (as  $T \rightarrow \infty$ ) capacity of the channel (20) over observations in the subspace  $\mathcal{H}_T = \mathcal{K}([0, T])$  of test functions with support in  $[0, T]$  is given by the expression (15) from the coding theorem above. Such a proof would be free from a simplification inherent in our model due to the assumed independence of the modes  $a_k$  for each  $T$ .

However an attempt to adapt the classical proof [4] meets obstacles arising from the additional symplectic structure and the fact that the observation subspace  $\mathcal{H}_T$  is *not invariant* under the complex structure  $J$ . Such kind of problems do not arise in the “narrowband” approximations of the type considered in [1] where the Planck vacuum spectrum is replaced by the flat one. A discussion of such a noise model is given in the next section.

### 3.2 Bandpass noise model

We mentioned that in the classical prototype of the gauge-covariant models 1-6,  $X, Y, Z$  are complex Gaussian random variables having circular distribution. This suggests to consider the following quantum noise model

$$\hat{\mathbf{Z}}(t) = \int_{-\infty}^{\infty} e^{i\omega t} d\hat{A}(\omega),$$

where  $\hat{A}(\omega)$ ,  $\omega \in \mathbb{R}$ , the quantum Gaussian independent increment process with the commutators (17) and correlation (18), but on the whole real line, with the spectral density  $N(\omega) \geq 0$ ,  $\omega \in \mathbb{R}$ . The noise has causal commutator

$$[\hat{\mathbf{Z}}(t), \hat{\mathbf{Z}}^{\dagger}(s)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-s)} d\omega = \delta(t-s). \quad (22)$$

The noise is thus generalized operator-valued process.

Then the *normally ordered* correlation function of the noise is

$$\beta(\mathbf{t} - \mathbf{s}) = \langle \hat{\mathbf{Z}}^\dagger(s) \hat{\mathbf{Z}}(t) \rangle = \int_{-\infty}^{\infty} e^{i\omega(t-s)} N(\omega) \frac{d\omega}{2\pi}.$$

Introducing the models 1-3 of gauge-covariant channels with filtered signal in the frequency domain by the relation (19) for all real  $\omega$  we arrive to the “time-invariant” model considered by Bardhan and Shapiro in [1] for a special noise spectral density  $N(\omega)$ .

A formal solution for the asymptotic capacity of such model is given by the relation similar to (15) but without  $\hbar\omega$ , namely:

$$C = \int (g(\bar{N}_\theta(\omega)) - g(N(\omega)))_+ \frac{d\omega}{2\pi},$$

where

$$\bar{N}_\theta(\omega) = \frac{1}{e^{\theta|K(\omega)|^{-2}} - 1}$$

and  $\theta$  is determined from

$$\int |K(\omega)|^{-2} (\bar{N}_\theta(\omega) - N(\omega))_+ \frac{d\omega}{2\pi} = E.$$

Notice that the capacity may be infinite unless  $|K(\omega)|$  decreases fast enough as  $\omega \rightarrow \infty$  (see Appendix 3). We anticipate that a general proof for this model can be given along the same lines as in the classical case [4] due to the special form of the commutator (22) which agrees with the simple complex structure of multiplication by  $i$ . Indeed, the relevant symplectic space is  $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}_+)$ , considered as *real* vector space with the skew-symmetric form and the vacuum inner product, correspondingly,

$$\begin{aligned} \Delta(f, g) &= \text{Im} \int_0^\infty \overline{f(t)} g(t) dt = (2\pi)^{-1} \text{Im} \int_{-\infty}^\infty \overline{\tilde{f}(\omega)} \tilde{g}(\omega) d\omega \\ j(f, g) &= \text{Re} \int_0^\infty \overline{f(t)} g(t) dt = (2\pi)^{-1} \text{Re} \int_{-\infty}^\infty \overline{\tilde{f}(\omega)} \tilde{g}(\omega) d\omega, \end{aligned}$$

and  $J$  is just multiplication by  $i$ . The subspace  $\mathcal{H}_T = L^2_{\mathbb{C}}([0, T])$  is *invariant* under  $J$ . Therefore the argument can be essentially a complexified version using the spectral theory for operators with Hermitian (rather than real symmetric) continuous kernels on  $[0, T]$  for the decomposition into normal modes, and the corresponding Kahrnen-Loewe expansion.

As argued in [1], such an approach is suitable for narrowband channels. A question that arises naturally is a derivation of this bandpass model from the broadband model of Sec. 3.1 under certain precise limiting conditions.

### 3.3 The symplectic eigenvalue problem

In the finite-dimensional case the normal mode decomposition is closely related to finding symplectic eigenvalues of the correlation matrix  $\alpha$ , which can be defined as numbers  $\lambda$  satisfying

$$[\alpha - i\lambda\Delta]f = 0$$

for some  $f \neq 0$ . The continuous-time analog of this is the integral equation

$$[\alpha_T - i\lambda\Delta_T]f = 0, \quad (23)$$

where  $\alpha_T$  is the integral operator with the symmetric kernel  $\alpha(t-s) = \beta(t-s) + \frac{1}{2}j(t-s)$  on  $[0, T]$ , and  $f \neq 0$  belongs to certain completion of the space  $K_T$ .

Next the problem arises to show that for  $T \rightarrow \infty$  the symplectic eigenvalues tend to the continuous spectral distribution.

The difference between the two models appears here most apparently:

Bandpass model: by complexification, (23) reduces to ordinary eigenvalue problem for the continuous hermitean kernel  $\beta(t-s) = \int_{-\infty}^{\infty} e^{i\omega(t-s)} N(\omega) \frac{d\omega}{2\pi}$ :

$$\int_0^T \beta(t-s)f(s)ds = (\lambda - 1/2)f(t), \quad t \in [0, T];$$

The limit  $T \rightarrow \infty$  can be treated as in the Kac-Merdock-Szöge theorem.

Broadband model: the symplectic eigenvalue equation (23) takes the form

$$\int_0^T \beta(t-s)f(s)ds + \frac{1}{2\pi} \int_0^T \frac{2f(t) - f(t-s) - f(t+s)}{s^2} ds = i\lambda f'(t), \quad t \in [0, T].$$

The study of such an equation is a subject of a future work.

## Appendix 1

We denote by  $S(\rho) = -\text{Tr}\rho \log \rho$  the von Neumann entropy. Let  $H$  be a positive selfadjoint operator, representing the energy. The constrained  $\chi$ -capacity of a channel  $\Phi$  can be expressed as [15]

$$C_\chi(\Phi; H, E) = \sup_{\text{tr } \bar{\rho}_\pi H \leq E} \left\{ S(\Phi[\bar{\rho}_\pi]) - \int S(\Phi[\rho])\pi(d\rho) \right\}, \quad (24)$$

where the maximization is performed over the set of input ensembles  $\pi$  (probability distributions on the set of quantum states (density operators)  $\rho$ ) satisfying the constraint  $\text{tr } \bar{\rho}_\pi H \leq E$  where  $\bar{\rho}_\pi = \int \rho \pi(d\rho)$  is the average state of the ensemble.

Let  $\Phi$  be an  $s$ -mode Gaussian gauge-covariant channel defined by the matrix parameters  $\mathbf{K}, \boldsymbol{\mu}$  as in [5], and  $H = \sum \epsilon_{kl} a_k^\dagger a_l$  a quadratic gauge-invariant Hamiltonian with Hermitean energy matrix  $\boldsymbol{\nu} = [\epsilon_{kl}]$ . By the solution of the quantum Gaussian optimizers conjecture [5], the quantity  $C_\chi(\Phi; H, E)$  is given by

$$\begin{aligned} & C_\chi(\Phi; H, E) \\ &= \max_{\boldsymbol{\nu}: \text{tr } \boldsymbol{\nu} \boldsymbol{\epsilon} \leq E} \text{tr } g(\mathbf{K}^* \boldsymbol{\nu} \mathbf{K} + \boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B)/2) - \text{tr } g(\boldsymbol{\mu} + (\mathbf{K}^* \mathbf{K} - \mathbf{I}_B)/2). \end{aligned} \quad (25)$$

The optimal ensemble  $\pi_*$  which attains the supremum in (24) consists of coherent states  $\rho_{\mathbf{x}}, \mathbf{x} \in \mathbb{C}^s$  distributed with gauge-invariant Gaussian probability distribution  $\pi_{\boldsymbol{\nu}}(d^{2s}x)$  on  $\mathbb{C}^s$  having zero mean and the correlation matrix  $\boldsymbol{\nu}$  which solves the maximization problem in (25). Here  $\rho_0$  is the vacuum density operator.

Next, let  $\Phi_k; k = 1, \dots, n$ , be gauge-covariant Gaussian channels,  $H_k$  the quadratic Hamiltonians. Put  $\Phi = \Phi_1 \otimes \dots \otimes \Phi_n$  and  $H = H_1 \otimes I \dots \otimes I + \dots + I \otimes \dots \otimes I \otimes H_n$ . Then the parameters  $\mathbf{K}, \boldsymbol{\mu}$  of  $\Phi$  have block-diagonal form. From (25) it follows that the maximizing  $\boldsymbol{\nu}$  also have the block-diagonal form and hence the additivity property follows

$$C_\chi(\Phi; H, E) = \max_{E_1 + \dots + E_n \leq E} [C_\chi(\Phi_1, H_1, E_1) + \dots + C_\chi(\Phi_n, H_n, E_n)]. \quad (26)$$

Consider the special case of independent uncorrelated noise modes, where  $\Phi_k$  is the one-mode channel in the canonical form given by the equation (13) with the energy operator  $H_k = \hbar \omega_k a_{k,X}^\dagger a_{k,X}$ . In this case  $E_k = \hbar \omega_k m_k$  and

$$C_\chi(\Phi_k, H_k, E_k) = g(|K(\omega_k)|^2 m_k + N(\omega_k)) - g(N(\omega_k))$$

so that (26) reduces to

$$C_\chi(\Phi; H, E) = \max \sum_k [g(|K(\omega_k)|^2 m_k + N(\omega_k)) - g(N(\omega_k))], \quad (27)$$

where the maximum is taken over the set

$$m_k \geq 0, \quad \sum_k \hbar \omega_k m_k \leq E.$$

Using the Kuhn-Tucker condition for the optimization of the concave function, which has the form

$$|K(\omega_k)|^2 g'(|K(\omega_k)|^2 m_k + N(\omega_k)) - \theta \hbar \omega_k \leq 0$$

with the equality if and only if  $m_k > 0$ , and taking into account that  $g'(x) = \log(1 + x^{-1})$ , one finds the “quantum water-filling” solution (cf. [10], [1])

$$\begin{aligned} C_\chi(\Phi; H, E) &= \sum_k [g(|K(\omega_k)|^2 m_k^* + N(\omega_k)) - g(N(\omega_k))] \\ &= \sum_k \left[ g \left( \frac{1}{e^{\theta \hbar \omega_k / |K(\omega_k)|^2} - 1} \right) - g(N(\omega_k)) \right]_+, \end{aligned} \quad (28)$$

where

$$m_k^* = |K(\omega_k)|^{-2} \left( \frac{1}{e^{\theta \hbar \omega_k / |K(\omega_k)|^2} - 1} - N(\omega_k) \right)_+,$$

and  $\theta$  is chosen in such a way that

$$\sum_k \hbar \omega_k m_k^* = E.$$

## Appendix 2. Proof of the Coding Theorem

We first prove the weak converse:

$$\inf_{\Sigma, M} \bar{\lambda}_T(\Sigma, M) \not\rightarrow 0 \quad \text{for } R > C. \quad (29)$$

From the classical Fano inequality and the quantum bound for classical information

$$TR \cdot (1 - \inf_{\Sigma, M} \bar{\lambda}_T(\Sigma, M)) \leq C_{\chi, T} + 1, \quad (30)$$

where  $C_{\chi, T}$  is the constrained  $\chi$ -capacity (24) of the channel  $\Phi_T$  in  $\mathcal{H}_T$ , so that  $C_{\chi, T} = C_\chi(\Phi_T; H, ET)$  given by (28).



Taking into account that

$$\Delta\omega = \frac{2\pi}{T}, \quad (31)$$

the energy constraint can be rewritten as

$$\sum_k \hbar\omega_k m_k \frac{\Delta\omega}{2\pi} \leq E.$$

The “quantum water-filling” solution is then

$$\frac{C_{\chi,T}}{T} = \sum_k [g(|K(\omega_k)|^2 m_k^* + N(\omega_k)) - g(N(\omega_k))] \frac{\Delta\omega}{2\pi}, \quad (32)$$

where

$$m_k^* = |K(\omega_k)|^{-2} \left( \frac{1}{e^{\theta_T f(\omega_k)} - 1} - N(\omega_k) \right)_+, \quad (33)$$

and

$$f(\omega) = \hbar\omega/|K(\omega)|^2 \geq \frac{\hbar\omega}{\kappa^2}, \quad (34)$$

while  $\theta_T$  is chosen in such a way that

$$\sum_k \hbar\omega_k m_k^* \frac{\Delta\omega}{2\pi} = E. \quad (35)$$

By considering the piecewise constant functions

$$N_T(\omega) = N(\omega_k), \quad K_T(\omega) = K(\omega_k), \quad f_T(\omega) = f(\omega_k), \quad \omega_{k-1} < \omega \leq \omega_k, \quad k = 0, 1, \dots$$

and

$$m_T(\omega) = m_k^* \quad \omega_{k-1} < \omega \leq \omega_k \leq \bar{\omega}(T); \quad m_T(\omega) = 0, \quad \omega > \bar{\omega}(T),$$

we can write the right hand side of (32) as

$$\begin{aligned} & \int_0^\infty [g(N_T(\omega) + |K_T(\omega)|^2 m_T(\omega)) - g(N_T(\omega))] \frac{d\omega}{2\pi} \\ &= \int_0^\infty [g(N(\omega) + |K(\omega)|^2 m_T(\omega)) - g(N(\omega))] \frac{d\omega}{2\pi} \end{aligned}$$

$$+ \int_0^\infty [g(N_T(\omega) + |K_T(\omega)|^2 m_T(\omega)) - g(N(\omega) + |K(\omega)|^2 m_T(\omega)) + g(N(\omega) - g(N_T(\omega)))] \frac{d\omega}{2\pi}. \quad (36)$$

Taking into account that

$$\int_0^\infty \hbar \omega m_T(\omega) \frac{d\omega}{2\pi} \leq \sum_k \hbar \omega_k m_k^* \frac{\Delta\omega}{2\pi} = E, \quad (37)$$

we see that the first term in the right hand side of (36) is less than or equals to

$$\max_{m \in \mathcal{M}} \int_0^\infty [g(N(\omega) + |K(\omega)|^2 m(\omega)) - g(N(\omega))] \frac{d\omega}{2\pi},$$

where

$$\mathcal{M} = \{m(\cdot) : m(\omega) \geq 0, \quad \int_0^\infty \hbar \omega m(\omega) \frac{d\omega}{2\pi} \leq E\}.$$

Similarly to (33), the solution is given by the function

$$m^*(\omega) = |K(\omega)|^{-2} \left( \tilde{N}_\theta(\omega) - N(\omega) \right)_+, \quad (38)$$

where  $\theta$  is determined from

$$\int_0^\infty \hbar \omega m^*(\omega) \frac{d\omega}{2\pi} = E,$$

in other words, the first term is less than or equals to  $C$ .

If we show that the second term in (36) tends to zero then we will have

$$\limsup_{T \rightarrow \infty} \frac{C_{\chi, T}}{T} \leq C \quad (39)$$

and therefore from (30)

$$(1 - \liminf_{T \rightarrow \infty} \inf_{\Sigma, M} \bar{\lambda}_T(\Sigma, M)) \leq C/R,$$

hence the weak converse (29).

We shall show it by using the Lebesgue dominated convergence theorem. Since  $N(\omega)$  is continuous,  $N_T(\omega) \rightarrow N(\omega)$  and  $g(N_T(\omega)) \rightarrow g(N(\omega))$  pointwise. Next we observe that  $\theta_T$  is separated from 0 as  $T \rightarrow \infty$ , that is

$\theta_T \geq \theta_\infty > 0$ . Indeed, assume that  $\theta_T \downarrow 0$  for some sequence  $T \rightarrow \infty$ , then the sequence of continuous functions

$$m_T(\omega) = |K_T(\omega)|^{-2} \left( \frac{1}{e^{\theta_T f_T(\omega)} - 1} - N_T(\omega) \right)_+ \quad (40)$$

converges to  $\infty$  uniformly in every interval  $0 < \underline{\omega} \leq \omega \leq \bar{\omega} < \infty$ , which contradicts to the condition (37). It follows that for any fixed  $\omega > 0$  the quantity

$$N_T(\omega) + |K_T(\omega)|^2 m_T(\omega) = \max \left( \frac{1}{e^{\theta_T f_T(\omega)} - 1}, N_T(\omega) \right)$$

is bounded as  $T \rightarrow \infty$ . Since  $g(x)$  is uniformly continuous on any bounded interval, it follows that

$$g(N_T(\omega) + |K_T(\omega)|^2 m_T(\omega)) - g(N(\omega) + |K(\omega)|^2 m_T(\omega)) \rightarrow 0$$

pointwise.

Let us show that the integrand is dominated by an integrable function. Taking into account that  $g'(x) \geq 0$  and  $g''(x) \leq 0$  for  $x \geq 0$ , we deduce that  $0 \leq g(x+y) - g(x) \leq g(y)$  for  $x, y \geq 0$ . Therefore the integrand is dominated from above by the function  $g(|K_T(\omega)|^2 m_T(\omega))$  and from below by the function  $-g(|K(\omega)|^2 m_T(\omega))$ . But from (40), (34)

$$|K_T(\omega)|^2 m_T(\omega) \leq \frac{1}{e^{\theta_T f_T(\omega)} - 1} \leq \frac{1}{e^{\theta_\infty f_T(\omega)} - 1} \leq \frac{1}{e^{c_\infty \omega} - 1} \quad (41)$$

with  $c_\infty = \theta_\infty \hbar / \kappa^2 > 0$ . Thus

$$g(|K_T(\omega)|^2 m_T(\omega)) \leq g \left( \frac{1}{e^{c_\infty \omega} - 1} \right) = \frac{c_\infty \omega}{e^{c_\infty \omega} - 1} - \log(1 - e^{-c_\infty \omega}),$$

which is positive integrable function. There is also a similar estimate from below. Thus (39) follows establishing the weak converse. The last inequality also implies that integrand in (15) is upperbounded by integrable function proving finiteness of the capacity, namely

$$C \leq \int_0^\infty g \left( \frac{1}{e^{c_\infty \omega} - 1} \right) \frac{d\omega}{2\pi} = \frac{\pi}{6c_\infty} = \frac{\pi \kappa^2}{6\hbar \theta_\infty}.$$

We now proceed to prove the direct statement of the coding theorem: for appropriately chosen codes the average error probability tends to zero when  $T \rightarrow \infty$  and  $R < C$ . Let us introduce some notations.

Denote  $\rho_{x_k} = |x_k\rangle\langle x_k|$ ;  $x_k \in \mathbb{C}$ , the coherent state for the  $k$ -th mode, and  $\rho_x = \otimes_k \rho_{x_k}$ ;  $x = \{x_k\}$  the coherent state for the collection of all modes with  $\omega_k \in [0, \bar{\omega}(T)]$ , so that the number of the modes is equal to  $s_T = \frac{\bar{\omega}(T)T}{2\pi}$ . In particular  $\rho_0$  is the vacuum state. We consider the codebooks of the form  $\Sigma = \{\rho_{x^1}, \dots, \rho_{x^N}\}$  and denote  $\rho'_j = \Phi_T[\rho_{x^j}]$ . It is Gaussian diagonal state with mean  $x^j = \{x_k^j\}$  and photon numbers  $\{N(\omega_k)\}$ .

Let  $\pi_*(d^{2s_T}x)$  be the Gaussian probability distribution

$$\pi_*(d^{2s_T}x) = \exp\left(-\sum_k \frac{|x_k|^2}{m_k^*}\right) \prod_k d^2x_k, \quad (42)$$

where  $m_k^*$  are given by (33). (If  $m_k^* = 0$ , we have in mind in (42) the Gaussian distribution degenerated at 0.)  $\pi_*$  is the optimal distribution on the coherent states on which  $C_{\chi, T}$  is achieved in (32).

Denote  $\bar{\rho}'_* = \Phi_T[\bar{\rho}_\pi]$ . It is Gaussian diagonal state with mean 0 and photon numbers

$$N'_k = |K(\omega_k)|^2 m_k^* + N(\omega_k) = \max\left\{\frac{1}{e^{\theta_T f(\omega_k)} - 1}, N(\omega_k)\right\}.$$

Define the suboptimal decoding  $M = \{M^1, \dots, M^N\}$  similarly to Eq. (44) in the proof of the coding theorem in [10]:

$$M^j = \left(\sum_{l=1}^N P P^l P\right)^{-\frac{1}{2}} (P P^j P) \left(\sum_{l=1}^N P P^l P\right)^{-\frac{1}{2}}, \quad (43)$$

where, however,  $P$  is the spectral projection of  $\bar{\rho}'_*$  corresponding to the eigenvalues in the range  $(e^{-[H(\bar{\rho}'_*) + \delta T]}, e^{-[H(\bar{\rho}'_*) - \delta T]})$ , and  $P^j$  is the spectral projection of  $\rho'_j$  corresponding to the eigenvalues in the range  $(e^{-[H(\rho_0) + \delta T]}, e^{-[H(\rho_0) - \delta T]})$ . Since  $\rho'_j$  are all unitarily equivalent to  $\rho'_0$ , then  $H(\rho'_j) = H(\rho'_0)$ , where  $\rho'_0 = \Phi_T[\rho_0]$  is Gaussian diagonal state with mean 0 and photon numbers  $\{N(\omega_k)\}$ .

Applying the basic inequality Eq. (50) from [10] with the word length  $n = 1$  and with  $\delta$  replaced by  $\delta T$ , we have

$$\inf_M \bar{\lambda}(\Sigma, M) \leq \quad (44)$$

$$\leq \frac{1}{N} \sum_{j=1}^N \left\{ 3\text{Tr}\rho'_j(I - P) + \text{Tr}\rho'_j(I - P_{x^j}) + \sum_{l \neq j} \text{Tr}P\rho'_j P P^l \right\},$$

Since  $\rho_j$  are unitary equivalent to  $\rho_0$ , then the middle term in (44) is simply

$$\text{Tr}\rho'_0(I - P_0), \quad (45)$$

which is similar to

$$\text{Tr}\bar{\rho}'_*(I - P). \quad (46)$$

We wish to estimate the terms (45), (46) for the Gaussian density operators  $\rho'_0, \bar{\rho}'_\pi$ . For definiteness let us take (45). We have

$$\text{Tr}\rho'_0(I - P_0) = \Pr \{ | -\log \lambda_{(\cdot)} - H(\rho_0) | \geq \delta T \}, \quad (47)$$

where  $\Pr$  is the distribution of eigenvalues  $\lambda_{(\cdot)}$  of  $\rho'_0$ . By Chebyshev inequality, this is less or equal to  $D(\log \lambda_{(\cdot)})/\delta^2 T^2$ . Now  $D(\log \lambda_{(\cdot)}) = \sum_k D_k(\log \lambda_{(\cdot)})$ , where  $D_k$  is the variance of  $\log \lambda_{(\cdot)}$  for the  $k$ -th mode. The eigenvalues of the Gaussian density operator  $\rho'_k(0)$  are

$$\lambda_n^k = \frac{N(\omega_k)^n}{(N(\omega_k) + 1)^{n+1}}; \quad n = 0, 1, \dots,$$

hence

$$D_k(\log \lambda_{(\cdot)}) = \sum_{n=0}^{\infty} (-\log \lambda_n^k - H(\rho_0))^2 \lambda_n^k \quad (48)$$

$$= \log^2 \frac{N(\omega_k) + 1}{N(\omega_k)} \sum_{n=0}^{\infty} (n - N(\omega_k))^2 \frac{N(\omega_k)^n}{(N(\omega_k) + 1)^{n+1}} = F(N(\omega_k)), \quad (49)$$

where

$$F(x) = x(x + 1) \log^2 \frac{x + 1}{x}$$

is a uniformly bounded function on  $(0, \infty)$ . Thus

$$\text{Tr}\rho_0(I - P_0) \leq \frac{\sum_k F(N(\omega_k))}{\delta^2 T^2} \leq \frac{c_1 s_T}{\delta^2 T^2} = \frac{c_2 \bar{\omega}(T)}{\delta^2 T}, \quad (50)$$

and a similar estimate holds for  $\text{Tr}\rho_\pi(I - P)$  with  $N(\omega_k)$  replaced by  $N'_k = N(\omega_k) + |K(\omega_k)|^2 m_k^*$ .

Let  $\mathbf{P}$  be a distribution on the set of  $N$  “words”  $x^1, \dots, x^N$ , under which the words are independent and have the probability distribution (42). Let

$$\nu_T = \mathbf{P}\left(\frac{1}{s_T} \sum_{k=1}^{s_T} \hbar\omega_k |x_k|^2 \leq E\right),$$

and remark that  $\mathbf{E} \frac{1}{s_T} \sum_{k=1}^{s_T} \hbar\omega_k |x_k|^2 \leq E$  (where  $\mathbf{E}$  is the expectation corresponding to  $\mathbf{P}$ ), hence by the central limit theorem

$$\lim_{T \rightarrow \infty} \nu_T \geq 1/2.$$

Let us explain why the central limit theorem holds for sums  $\sum_{k=1}^{s_T} \hbar\omega_k |x_k|^2$  as  $T \rightarrow \infty$ . The summands are squares of the normal random variables  $\xi_{k,T} = \sqrt{\hbar\omega_k} x_k$  which have zero means and the uniformly bounded variances (see (41))

$$\hbar\omega_k m_k^* \leq \frac{\hbar\omega_k}{|K(\omega_k)|^2} \frac{1}{e^{c_\infty f(\omega_k)} - 1} = c_\infty^{-1} \frac{c_\infty f(\omega_k)}{e^{c_0 f(\omega_k)} - 1} \leq c_\infty^{-1}.$$

Therefore the Liapunov condition is fulfilled ensuring convergence of the properly normalized sums  $\sum_{k=1}^{s_T} \xi_{k,T}^2$  to the normal distribution.

Define the modified distribution  $\tilde{\mathbf{P}}$  under which the words are still independent but have the distribution

$$\tilde{\pi}_*(d^{2s_T} x) = \begin{cases} \nu_T^{-1} \pi_*(d^{2s_T} x), & \text{if } \sum_{k=1}^{s_T} \hbar\omega_k |x_k|^2 \leq ET, \\ 0, & \text{otherwise.} \end{cases} \quad (51)$$

Therefore  $\tilde{\mathbf{E}}\xi \leq \nu_T^m \mathbf{E}\xi \leq 3^m \mathbf{E}\xi$  for any nonnegative random variable  $\xi$  depending on  $m$  words and  $T$  large enough.

Now let  $x^1, \dots, x^N$  be taken randomly with the joint probability distribution  $\tilde{\mathbf{P}}$ . Since the right hand side of (44) depends at most on  $m = 2$  words,

$$\begin{aligned} \tilde{\mathbf{E}} \inf_M \bar{\lambda}(\Sigma, M) &\leq \frac{1}{N} \sum_{j=1}^N \left\{ 9M \text{Tr} \rho_{x^{(j)}}(I - P) + \text{Tr} \rho_0(I - P_0) + \sum_{k \neq j} 9 \mathbf{E} \text{Tr} P \rho_{x^{(j)}} P P_{x^{(k)}} \right\} \\ &= 9 \text{Tr} \rho_\pi(I - P) + \text{Tr} \rho_0(I - P_0) + 9(N-1) e^{-(C_{\chi,T} - 2\delta T)} \\ &\leq \frac{c_3 \bar{\omega}(T)}{\delta^2 T} + 9 e^{(RT + 2\delta T - C_{\chi,T})}, \end{aligned}$$

where (50) was used to estimate the first two terms. To complete the proof we have only to show that

$$\liminf_{t \rightarrow \infty} \frac{C_{\chi, T}}{T} \geq C. \quad (52)$$

Let  $m^*(\omega)$  be the function (38), and let  $\omega'_k$  be the point on the segment  $[\omega_{k-1}, \omega_k]$  at which it achieves its minimum, then

$$\frac{1}{2\pi} \hbar \omega'_k m^*(\omega'_k) \leq \int_{\underline{\omega}}^{\tilde{\omega}} \hbar \omega m^*(\omega) \frac{d\omega}{2\pi} = E,$$

hence

$$\frac{C_{\chi, T}}{T} \geq \sum_{k=1}^{s_T} [g(N(\omega_k) + |K(\omega_k)|^2 m^*(\omega'_k)) - g(N(\omega_k))] \frac{\Delta\omega_k}{2\pi}.$$

Since  $N(\omega)$ ,  $K(\omega)$  and  $m^*(\omega)$  are continuous and the summand is nonnegative, the limit of the last sum is greater than or equals to

$$\int_0^{\tilde{\omega}} [g(N(\omega) + |K(\omega)|^2 m^*(\omega)) - g(N(\omega))] d\omega,$$

for any fixed  $\tilde{\omega} > 0$ . Letting  $\tilde{\omega} \uparrow \infty$  we obtain (52) and the proof is completed.

### Appendix 3. The infinite capacity

We have seen that in the *quantum broadband noise model* the capacity is finite as follows from the estimate of the Theorem in Sec.2.

Let us show that the capacity can be infinite in the *quantum bandpass noise model* (Subsec. 3.2). For simplicity we consider the case  $K(\omega) \equiv 1$ . Let  $N(\omega) \geq 0$  be the spectral density of the quantum noise and  $m(\omega)$  a spectral distribution of the signal. Then

$$C = \sup_{m \in \mathcal{M}} \int_0^\infty [g(N(\omega) + m(\omega)) - g(N(\omega))] \frac{d\omega}{2\pi},$$

where

$$\mathcal{M} = \{m(\cdot) : m(\omega) \geq 0, \quad \hbar \Omega \int_0^\infty m(\omega) \frac{d\omega}{2\pi} = E\}.$$

Here  $\Omega$  is the “carrier frequency”. Assume that  $N(\omega)$  is monotonously decreasing for  $\omega$  large enough and tends to 0 as  $\omega \rightarrow \infty$ . We will show that  $C = \infty$  by choosing rectangular  $m(\omega)$  such that

$$m(\omega) = \begin{cases} M & \text{if } \omega \in [\omega_1, \omega_2] \\ 0 & \text{otherwise} \end{cases},$$

where  $M = \frac{2\pi E}{\hbar\Omega(\omega_2 - \omega_1)}$ . We use the fact that if  $\phi(x)$  is concave increasing function (so that  $\phi'(x)$  is decreasing), then

$$\phi(x+y) - \phi(x) \geq \phi'(x+y)y, \quad x, y \geq 0. \quad (53)$$

Applying this for  $\phi(x) = g(x)$  and using the fact that  $g'(x) = \log(x+1) - \log x$  is decreasing function with  $g'(x) \geq -\log x$ , we obtain

$$\begin{aligned} C &\geq \int_{\omega_1}^{\omega_2} [g(N(\omega) + M) - g(N(\omega))] \frac{d\omega}{2\pi} \\ &\geq \int_{\omega_1}^{\omega_2} g'(N(\omega) + M) M \frac{d\omega}{2\pi} \\ &\geq g'(N(\omega_1) + M) M \frac{(\omega_2 - \omega_1)}{2\pi} \\ &= \frac{E}{\hbar\Omega} g'(N(\omega_1) + M) \\ &\geq -\frac{E}{\hbar\Omega} \log(N(\omega_1) + M). \end{aligned}$$

Choosing  $\omega_1 \rightarrow \infty$ ,  $\omega_2 - \omega_1 \rightarrow \infty$ , which amounts to  $\omega_1 \rightarrow \infty$ ,  $M \rightarrow 0$ , we obtain  $C = \infty$ .

Similarly, in the *classical case* we have

$$C = \frac{1}{2} \sup_{m \in \mathcal{M}} \int_0^\infty [\log(N(\omega) + m(\omega)) - \log(N(\omega))] \frac{d\omega}{2\pi},$$

where

$$\mathcal{M} = \{m(\cdot) : m(\omega) \geq 0, \quad \int_0^\infty m(\omega) \frac{d\omega}{2\pi} = E\}.$$

Then applying (53) to  $\phi(x) = \log x$  and using the fact that  $[\log x]' = x^{-1}$  is decreasing function we obtain

$$C \geq \frac{1}{2} \frac{E}{N(\omega_1) + M} \rightarrow \infty$$



as  $\omega_1 \rightarrow \infty$ ,  $\omega_2 - \omega_1 \rightarrow \infty$ .

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